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Geometrical effects in resonant gas oscillations

By H. OCKENDON, J. R. OCKENDON, M. R. PEAKE AND W. CHESTER

Mathematical Institute, University of Oxford, 24–29 St Giles, Oxford OX1 3LB, UK

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It is known that the response of a cylindrical acoustic resonator to excitation by an oscillating piston can contain shock waves if the detuning is sufficiently small. However, the response of a spherical annular resonator is continuous, with an amplitude that depends on the detuning in the same way as does a Duffing equation. This paper discusses the response in resonators that deviate from being cylindrical and shows that, in general, the detuning range in which shocks are possible decreases as the geometrical imperfection increases.

1. Introduction

The modern theory of nonlinear effects in one-dimensional gas oscillations was initiated by Chester (1964), who derived an ordinary differential equation for the periodic response near resonance of a cylindrical organ pipe. In the absence of dissipation, this model described the genesis of shock waves as the detuning (i.e. the difference between the natural and forcing frequencies) was progressively decreased, and it predicted a sharp cut-off above which no shock waves were present. It has since been the basis for many generalizations, in particular to the case of open-ended pipes, and to include damping and dispersion effects (Chester 1981).

More recently Chester (1991) has considered spherically symmetric oscillations in a sphere and shown that the effect of nonlinearity on the periodic response is quite different, being the same as that of a nonlinear spring, in that the response takes the form of a shock-free single mode whose amplitude satisfies a cubic equation in which the detuning appears as a coefficient, exactly as in Duffing's equation. The same response is found for waves in the space between concentric spheres as long as their separation is sufficiently large compared to the forcing amplitude (Peake 1993). This immediately poses the question of how this 'single-mode' response tends to the 'infinite-number of modes' response of the organ pipe when the annulus becomes thinner. As mentioned by Keller (1977), the excitation, via nonlinearity, of an infinite number of modes is crucially dependent on the linear response having an infinite number of commensurate natural frequencies, as in the case of the organ pipe; for the spherical case the spectrum is non-commensurate and hence has a single mode response. The principle concern of this paper is the way in which the removal of this degeneracy of the spectrum by increasing the geometric imperfection governs the response of an undamped acoustic resonator.

Degenerate spectra can also be removed by invoking the shallow-water analogy and by introducing the dispersive effects inherent in surface gravity waves on shallow water. This scenario was considered first by Chester (1968) and later by Ockendon, Ockendon & Johnson (1986), and a higher-order version of Chester's original equation (see also Chester & Bones 1968) was analysed to show that when any dispersive effects are introduced, no matter how small, the response is always shock free; however, the response comprises an infinite number of branches lying closer and closer to each other as the dispersion is progressively decreased, which presumably allows the possibility of chaotic responses. Conversely, as the dispersion is increased, fewer and fewer modes are found in any given detuning band near the fundamental and eventually a single-mode response emerges (again with a Duffing-like structure) (Moiseyev 1958). This is quite unlike the progressive increase of dissipative effects introduced by Chester (1968) and Keller (1976*a*) to model boundary layers on the walls of the organ pipe which only partially 'disperse' the shock response when they are small. There, once the viscosity is larger than a critical value, the response is smooth and eventually attains a 'single mode' response when the viscosity is large enough (Keller 1976*a*).

The aim of this paper is to try to present a simple scenario for geometrical variations. To minimize the distraction caused by excessive algebraic complications we begin by manipulating the equations of nearly unidirectional gas dynamics in a tube into a form (2.12) where the relevant asymptotic expansion can be carried out as economically as possible. The model we use is equivalent to the established quasi-one-dimensional equations (see, for instance, Lighthill 1978, Section 2.13) used by Keller (1977), Chester (1993) and Ellermeier (1993) but the derivation carried out in §2 for a two-dimensional tube quantifies the geometrical variations that are allowable if this model is to be used. In §3 these expansions are then performed in terms of a parameter ϵ characterizing the small forcing amplitude. The dimensionless detuning must be of $O(\epsilon^{\frac{1}{2}})$ if it is to interact with nonlinearity to produce an interesting response. The geometric variations can enter into the asymptotics at any one of three levels, and, in crude terms, we will find the following.

(i) For geometric variations much smaller than $\epsilon^{\frac{1}{2}}$, the response is governed by Chester's (1964) ordinary differential equation.

(ii) For geometric variations of $O(\epsilon^{\frac{1}{2}})$, the response is described by an integro-differential equation for the waveform. This equation can still support shock waves when the integral term introduced by the area variations is small enough but, beyond a critical size, a smooth response is predicted except for resonators of special shape.

(iii) For geometric variations much larger than $\epsilon^{\frac{1}{2}}$, the response is more difficult to elucidate but we suggest that shock waves are only ever present for very special area variations.

Hence our conclusion will be that, in general, shocks can only occur in the response when either the linearized spectrum contains an infinite number of commensurate frequencies or the amplitude of the geometric imperfection is $O(\epsilon^{\frac{1}{2}})$ or less. Also, the 'less commensurate' is the spectrum or the larger the geometric imperfection, the more likely we are to see a single-mode response. Thus as far as shock waves are concerned, the effects of geometry can be likened to those of detuning or boundary-layer damping: in most cases there is a critical 'cut-off' value of the imperfection above which the response is smooth. However, there do exist classes of resonators for which the response contains shocks even for large imperfections.

We note that our work should have implications for weakly nonlinear *free* oscillations of gas in imperfect resonators in the same way that the results of Keller & Ting (1966) can be derived from those of Chester (1964) by letting the response amplitude tend to infinity. We will consider this briefly in the conclusion.

We now set up our model under the assumption that the resonator is two-dimensional. This is done purely for ease of exposition, our model (2.12), (2.13) being valid for nearly cylindrical resonators of arbitrary cross-section.

2. The model

Beginning with purely two-dimensional flow in a nearly rectangular resonator, we seek the $2\pi/\omega$ time-periodic response of the following dimensional equation (see Ockendon & Tayler 1983, Ch. 4) for the velocity potential ϕ:

$$(a^2-\phi_x^2)\,\phi_{xx}-2\phi_x\,\phi_y\,\phi_{xy}+(a^2-\phi_y^2)\,\phi_{yy}=\frac{\partial}{\partial t}\{\phi_t+|\nabla\phi|^2\}, \tag{2.1}$$

where the speed of sound

$$a=\{a_0^2-(\gamma-1)\,(\phi_t+\tfrac{1}{2}|\nabla\phi|^2)\}^{\frac{1}{2}} \tag{2.2}$$

and we are assuming that any shocks that are generated are so weak that entropy changes across them can be neglected. This incurs an error of $O(|\nabla\phi|/a_0)^3$, and we will consider the implications of this constraint in the conclusion. Assuming symmetry about the x-axis, the boundary conditions are

$$\left.\begin{aligned}\phi_x&=0\quad\text{at}\quad x=0,\\ \phi_x&=l\omega\sin\omega t\quad\text{at}\quad x=\pi L-l\cos\omega t,\\ \phi_y&=0\quad\text{at}\quad y=0;\end{aligned}\right\} \tag{2.3}$$

and

$$\phi_y=\frac{h_0}{L}\sigma h'\left(\frac{x}{L}\right)\phi_x\quad\text{on}\quad y=h_0\left(1+\sigma h\left(\frac{x}{L}\right)\right), \tag{2.4}$$

where the resonator length πL, breadth h_0 and slope $\sigma h_0/L$ specify the geometry, and l,ω specify the amplitude and frequencies of the driving mechanism. Note that we have assumed that area variations are on the lengthscale of the tube. Non-dimensionalizing time with ω^{-1}, x with L, y with h_0, a with a_0 and ϕ with $lL\omega$, and writing

$$\epsilon=l/L,\ \delta=(L\omega/a_0)^2-1$$

we obtain

$$\begin{aligned}\frac{L^2}{h_0^2}\phi_{yy}+\phi_{xx}-(1+\delta)\,\phi_{tt}&=\epsilon(1+\delta)\left[2\phi_x\,\phi_{xt}+(\gamma-1)\,\phi_t\left(\phi_{xx}+\frac{L^2}{h_0^2}\phi_{yy}\right)+2\frac{L^2}{h_0^2}\phi_{yt}\,\phi_y\right]\\ &\quad+\epsilon^2(1+\delta)\left[\frac{\gamma+1}{2}\left(\phi_x^2\,\phi_{xx}+\frac{L^4}{h_0^4}\phi_y^2\,\phi_{yy}\right)+\frac{(\gamma-1)}{2}\frac{L^2}{h_0^2}(\phi_y^2\,\phi_{xx}+\phi_x^2\,\phi_{yy})+2\frac{L^2}{h_0^2}\phi_x\,\phi_y\,\phi_{xy}\right],\end{aligned} \tag{2.5}$$

with

$$\phi_x=0\quad\text{at}\quad x=0,\quad \phi_x=\sin t\quad\text{at}\quad x=\pi-\epsilon\cos t, \tag{2.6}$$

$$\phi_y=0\quad\text{on}\quad y=0,\quad \phi_y=\frac{\sigma h_0^2}{L^2}h'(x)\,\phi_x\quad\text{on}\quad y=1+\sigma h(x), \tag{2.7}$$

and 2π-periodicity in time.

The dimensionless parameters are σ and h_0/L which characterize the geometry and which are assumed to be $\leqslant O(1)$, ϵ which is the small forcing amplitude, and δ which measures the detuning of the forcing frequency from the fundamental frequency of the rectangle. It is easily seen that when $\epsilon=\sigma=0$, the amplitude of the response is of $O(\delta^{-1})$ as $\delta\to 0$. The detuning regime of greatest interest is where the nonlinear, forcing and detuning effects are comparable, and, as shown in Chester (1964), this happens when $\lambda=\delta\epsilon^{-\frac{1}{2}}=O(1)$ as $\epsilon\to 0$. This comes about because the quadratic nonlinearity can only affect the lowest-order response when $\phi \sim O(\epsilon^{-\frac{1}{2}})$ and hence we immediately scale $\phi = \epsilon^{-\frac{1}{2}}\Phi$ to give

$$\frac{L^2}{h_0^2}\Phi_{yy}+\Phi_{xx}-\Phi_{tt} = \epsilon^{\frac{1}{2}}\left(\lambda\Phi_{tt}+2\Phi_x\Phi_{xt}+(\gamma-1)\Phi_t\Phi_{xx} + 2\frac{L^2}{h_0^2}\Phi_y\Phi_{yt}+(\gamma-1)\frac{L^2}{h_0^2}\Phi_t\Phi_{yy}\right), \quad (2.8a)$$

plus smaller terms, with

$$\left.\begin{aligned} \Phi_x = 0 \quad \text{on} \quad x=0, \quad \Phi_x = \epsilon^{\frac{1}{2}}\sin t \quad \text{on} \quad x = \pi + O(\epsilon), \\ \Phi_y = 0 \quad \text{on} \quad y=0, \quad \Phi_y = \sigma\frac{h_0^2}{L^2}h'(x)\Phi_x \quad \text{on} \quad y = 1+\sigma h(x). \end{aligned}\right\} \quad (2.8b)$$

If we are away from resonance we expect to be able to find the linear response in the form of a perturbation about a one-dimensional solution as long as either σ or h_0/L is small and we now attempt to quantify the quasi-one-dimensional approximation more precisely. Since the y-variation is imposed by boundary condition (2.8b) we can write

$$\phi = \bar{\phi}(x,t)+\frac{\sigma h_0^2}{L^2}\hat{\phi}(x,y,t), \quad (2.9a)$$

where
$$\int_0^{1+\sigma h}\hat{\phi}\,\mathrm{d}y = 0. \quad (2.9b)$$

The boundary condition at $x=\pi$ implies that we can only obtain a quasi-one-dimensional solution correct to $O(\epsilon^{\frac{1}{2}})$ if

$$\sigma h_0^2/L^2 \ll \epsilon^{\frac{1}{2}} \quad (2.10)$$

and then the end conditions will be $\bar{\phi}_x = 0$ at $x=0$ and $\bar{\phi}_x = \epsilon^{\frac{1}{2}}\sin t$ at $x=\pi$. Equation (2.8) becomes

$$\sigma\hat{\phi}_{yy}+\bar{\phi}_{xx}-\bar{\phi}_{tt}+\sigma\frac{h_0^2}{L^2}(\hat{\phi}_{xx}-\hat{\phi}_{tt}) = \epsilon^{\frac{1}{2}}[\lambda\bar{\phi}_{tt}+2\bar{\phi}_x\bar{\phi}_{xt}+(\gamma-1)\bar{\phi}_t\bar{\phi}_{xx} + (\gamma-1)\sigma\bar{\phi}_t\hat{\phi}_{yy}]+O\left(\epsilon^{\frac{1}{2}}\sigma\frac{h_0^2}{L^2}\right), \quad (2.11a)$$

with $\hat{\phi}_y = h'(x)\left(\bar{\phi}_x+\sigma\frac{h_0^2}{L^2}\hat{\phi}_x\right)$ on $y = 1+\sigma h$, $\quad \hat{\phi}_y = 0$ on $y=0$. (2.11b)

When we integrate (2.11a) across the resonator from $y=0$ to $y=1+\sigma h$, and use (2.11b), we find that

$$((1+\sigma h)\bar{\phi}_x)_x-(1+\sigma h)\bar{\phi}_{tt} = \epsilon^{\frac{1}{2}}(\lambda(1+\sigma h)\bar{\phi}_{tt}+2(1+\sigma h)\bar{\phi}_x\bar{\phi}_{xt}+(\gamma-1)\bar{\phi}_t((1+\sigma h)\bar{\phi}_x)_x) \quad (2.12)$$

is correct to $O(\epsilon^{\frac{1}{2}})$ as long as (2.10) holds. We can therefore work with this quasi-one-dimensional model which will satisfy (2.12) subject to

$$\bar{\phi}_x(0,t) = 0, \bar{\phi}_x(\pi,t) = \epsilon^{\frac{1}{2}}\sin t. \quad (2.13)$$

In summary we have shown that we can use (2.12) to consider $O(1)$ area variations

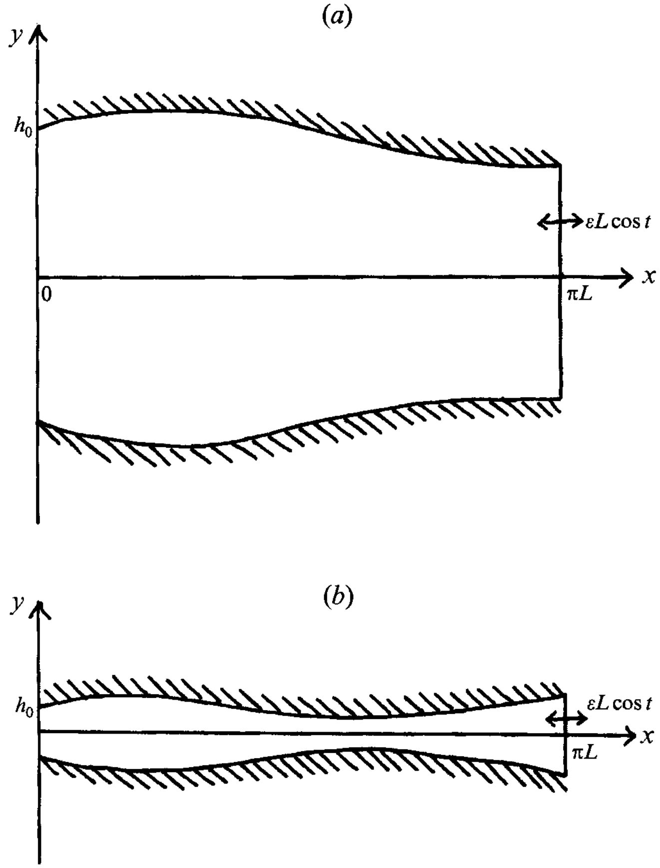

FIGURE 1. Possible geometries: (a) 'thick' resonator with $\sigma \ll \epsilon^{\frac{1}{2}}$, $h_0/L = O(1)$; (b) 'thin' resonator with $\sigma = 1$, $h_0^2/L^2 \ll \epsilon^{\frac{1}{2}}$.

for a 'thin' tube $(h_0^2/L^2 \ll \epsilon^{\frac{1}{2}})$, but we can only consider $o(\epsilon^{\frac{1}{2}})$ variations in the area of a 'thick' tube for which $h_0^2/L^2 = O(1)$ (figure 1). Also we note that exactly the same analysis is possible for nearly cylindrical resonators of arbitrary cross-section defined by $F(y/h_0, z/h_0, \sigma h(x/L)) = 0$, where $h_0^2(1+\sigma h(x/L))$ is the area of the cross-section and again we find that (2.12) is valid whenever $\sigma h_0^2/L^2 \ll \epsilon^{\frac{1}{2}}$.

We will now analyse the solution of (2.12) for different values of σ. We start with $\sigma = O(\epsilon^{\frac{1}{2}})$, when the effect of the geometric variation and the nonlinearity are comparable and we will then increase σ in an attempt to describe the transition from the shock regime of Chester (1964), which corresponds to $\sigma = 0$, to the single-mode response which is expected when $\sigma = O(1)$.

3. Weak geometric effects: the analysis of (2.12) for $\sigma = O(\epsilon^{\frac{1}{2}})$

We write $\sigma = \kappa\epsilon^{\frac{1}{2}}$, where $\kappa \leqslant O(1)$, and expanding the solution of (2.12) in the form

$$\bar{\phi} \sim \phi_0(x,t) + \epsilon^{\frac{1}{2}}\phi_1(x,t) + \dots \tag{3.1}$$

gives

$$\phi_{0xx} - \phi_{0tt} = 0 \quad \text{with} \quad \phi_{0x} = 0 \quad \text{at} \quad x = 0, \pi \tag{3.2}$$

and

$$\phi_{1xx} - \phi_{1tt} = \lambda\phi_{0tt} + 2\phi_{0x}\phi_{0xt} + (\gamma-1)\phi_{0t}\phi_{0xx} - \kappa h'\phi_{0x}, \tag{3.3a}$$

with

$$\phi_{1x}(0,t) = 0, \quad \phi_{1x}(\pi,t) = \sin t, \quad \phi_1(x,t) = \phi_1(x, t+2\pi). \tag{3.3b}$$

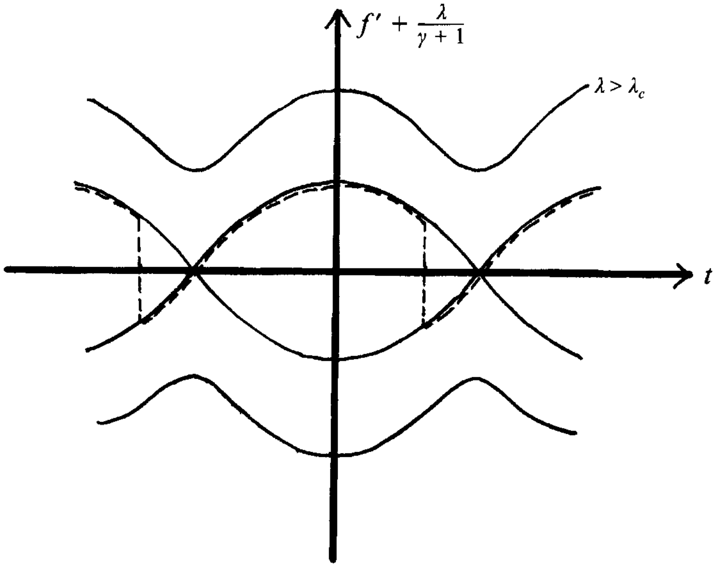

FIGURE 2. Solutions of (3.7): ——, $|\lambda| > \lambda_c$; ----, $|\lambda| < \lambda_c$.

Solving (3.2) gives $\phi_0 = f(\xi)+f(\eta)$ where $\xi = t+x$ and $\eta = t-x$ and f has period 2π, and then ϕ_1 satisfies

$$4\phi_{1\xi\eta} = -\lambda(f''(\xi)+f''(\eta))-(\gamma+1)(f'(\xi)f''(\xi)+f'(\eta)f''(\eta)) \\ -(\gamma-3)(f'(\xi)f''(\eta)+f'(\eta)f''(\xi))+\kappa h'((\xi-\eta)/2)(f'(\xi)-f'(\eta)). \quad (3.4)$$

Hence, since (3.3b) implies that

$$\int_t^{t+2\pi} \phi_{1\xi\eta}|_{\eta=t}\,\mathrm{d}\xi + \int_t^{t+2\pi} \phi_{1\xi\eta}|_{\xi=t+2\pi}\,\mathrm{d}\eta = \sin t, \quad (3.5)$$

the solvability condition for ϕ_1 is that $f(t)$ should satisfy

$$\lambda f'' + (\gamma+1)f'f'' + \frac{1}{\pi}\sin t = \frac{\kappa}{2\pi}\int_0^{\pi} (h'(\tau)-h'(\pi-\tau))f'(t+2\tau)\,\mathrm{d}\tau, \quad (3.6)$$

and we choose f to have zero mean over a period.

This equation was derived by Keller (1977) and more recently by Ellermeier (1993), who has shown that it also applies to a uniform tube containing an initially stratified density profile. When $\kappa = 0$, (3.6) reduces to Chester's equation

$$\lambda f'' + (\gamma+1)f'f'' + \frac{1}{\pi}\sin t = 0 \quad (3.7)$$

which can be solved exactly. The solution is summarized in figure 2 and the corresponding response diagram is shown in figure 4(a). When $|\lambda| \geqslant \lambda_c = 4[(\gamma+1)/\pi^3]^{\frac{1}{2}}$, there is a continuous solution, but for $|\lambda| < \lambda_c$ the periodicity condition can only be satisfied by introducing a jump discontinuity at $t = t^* = 2\sin^{-1}(\lambda/\lambda_c)$. This jump is defined uniquely if we insist that the shock to which it corresponds is compressive. The physical quantities all satisfy the Rankine–Hugoniot weak shock conditions across the discontinuity which travels with speed a_0 up and down the tube and is reflected without change of strength at the ends $x = 0, \pi$.

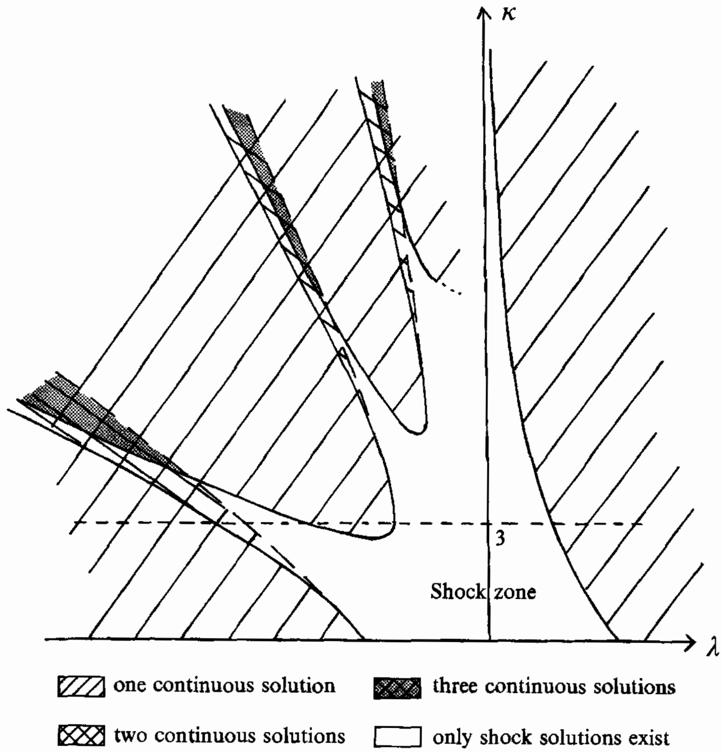

FIGURE 3. Sketch of solutions of (3.8) in κ–λ space.

We now consider the effect of the integral term in (3.6). Chester (1964) showed that the effect of viscous boundary layers resulted in an extra term similar in form to the right-hand side of (3.6) but with the expression $h'(\tau)-h'(\pi-\tau)$ replaced by a function which is *not* odd about $\tau = \frac{1}{2}\pi$. The crucial distinction between the two effects is apparent when we repeatedly integrate the right-hand side of (3.6) by parts. We see that we only generate *even* derivatives of f and hence our area variations are inevitably dispersive; the kernel engendered by the boundary layer leads to odd derivatives of f and hence is dissipative.

It can be seen immediately that (3.6) reduces to (3.7) if $h'(x)$ is constant and also whenever $h'(x) = h'(\pi-x)$. In order to understand the response in a non-trivial case we start by discussing a special case for which (3.6) reduces to an ordinary differential equation. This occurs when $h' = 2x$ and (3.6) becomes

$$\lambda f'''(t)+(\gamma+1)f'(t)f''(t)+\frac{1}{\pi}\sin t = \kappa f(t). \tag{3.8}$$

This equation can be studied both asymptotically, for small and large κ, and numerically and the results are summarized in figures 3 and 4. In figure 3, the shaded areas of the (λ,κ)-plane indicate values for which continuous 2π-periodic solutions exist. Only values of $\kappa > 0$ are shown since replacing κ, λ and $f(t)$ by their negative values leads to exactly the same problem. In the unshaded regions it is necessary to introduce a shock or shocks to get a solution for f that satisfies the periodicity condition. The corresponding response curves for the various values of κ are sketched in figure 4 where the shock regime is indicated by broken lines. This figure uses

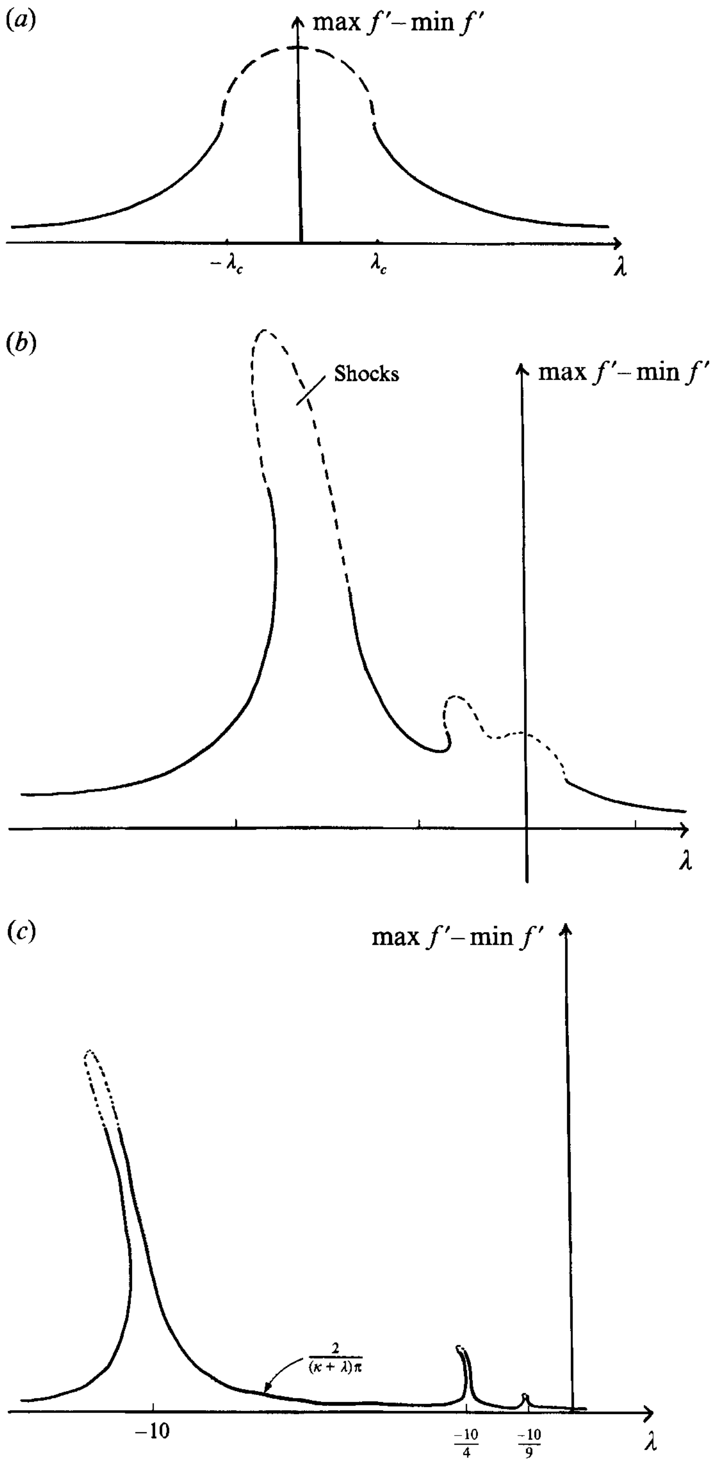

FIGURE 4. Response diagrams for solutions of (3.8): (*a*) $\kappa = 0$; (*b*) $\kappa = 3$; (*c*) $\kappa = 10$.

information obtained from numerical solutions to show how the solution evolves from that of (3.7) when $\kappa = 0$ into truncated Duffing-like response curves as κ increases. As κ increases and the response curves 'turn over', there are values of λ for which two possible smooth solutions coexist with a shock solution and eventually there are also values for which three possible smooth solutions coexist as shown in figure 3. It can clearly be seen that the number of 'resonant branches' increases with κ and the range of λ for which no continuous solutions exist decreases with κ.

As $\kappa \rightarrow \infty$ there are Duffing-like superharmonic 'resonances' of frequency N where $\lambda \approx -\kappa/N^2$ and, away from these values, $f \approx (\sin t)/\pi(\kappa+\lambda)$. We now sketch the asymptotic analysis near the first 'resonance' at $\lambda = -\kappa$. Writing $\lambda = -\kappa + \lambda_1 \kappa^{-\frac{1}{3}}$ and $f = \kappa^{\frac{1}{3}} \bar{f}$, equation (3.8) is

$$\bar{f}'' + \bar{f} = (\gamma+1)\, \kappa^{-\frac{2}{3}} \bar{f}' \bar{f}'' + \kappa^{-\frac{4}{3}} \left(\lambda_1 \bar{f}'' + \frac{1}{\pi} \sin t \right).$$

Now writing
$$\bar{f} = f_0 + \kappa^{-\frac{2}{3}} f_1 + \kappa^{-\frac{4}{3}} f_2 + \ldots$$
we find
$$f_0 = A \sin t$$
and
$$f_1 = \tfrac{1}{6}(\gamma+1)\, A^2 \sin 2t,$$
so that
$$f_2'' + f_2 = (\gamma+1)(f_0' f_1'' + f_1' f_0'') + \lambda_1 f_0'' + \frac{\sin t}{\pi}$$
and f_2 is periodic only if A satisfies the equation

$$-\tfrac{1}{6}(\gamma+1)^2 A^3 - \lambda A + \frac{1}{\pi} = 0,$$

which gives rise to a response of the type sketched in figure 4(*c*). On the large-amplitude branches, $\lambda \approx -\frac{1}{6} A^2 (\gamma+1)^2$, and their structure as $A \rightarrow \infty$ can be investigated further by writing $\lambda = -\kappa\alpha$ and $f = \kappa \hat{f}$ to obtain

$$\alpha \hat{f}'' + \hat{f} - (\gamma+1)\, \hat{f}' \hat{f}'' = \frac{1}{\pi \kappa^2} \sin t.$$

Expanding $\hat{f} = \hat{f}_0 + 1/\kappa^2 \hat{f}_1 + \ldots$ and $\alpha = \alpha_0 + \kappa^{-2} \alpha_1 + \ldots$ now leads to an autonomous equation for $\hat{f}_0$ with an undetermined phase. The phase plane for $\hat{f}_0$ is sketched in figure 5 and, since it can be shown that the periodic orbits have periods lying between $6\alpha_0^{\frac{1}{2}}$ and $2\pi\alpha_0^{\frac{1}{2}}$, solutions of period 2π are only possible for $1 \leqslant \alpha_0 \leqslant \frac{1}{9}\pi^2$. The integrability condition for the equation for $\hat{f}_1$ determines two possible phases corresponding to the two large-amplitude branches in figure 4(*c*). It can also be inferred that these two branches are separated in amplitude by $O(1/\kappa^2)$ and that they terminate when $\alpha \approx \frac{1}{9}\pi^2 \pm \alpha_c/\kappa^2$ where $\alpha_c = 1.615\ldots$. Numerical work indicates that solutions containing shocks where $\hat{f}_0'$ is discontinuous are possible on the dotted curves in figure 4(*c*).

A similar analysis can be performed for $\lambda \simeq -\kappa/N^2$ for $N > 1$; the magnitude of f in the 'Duffing-like' region is $O(\kappa^{-\frac{1}{3}})$ when $N = 2$ and the amplitude at the end of the branches is κ/N^2 as shown in figure 4(*c*) (Can & Askar 1990 and Peake 1993). No work has been done on the stability of these solutions.

Other geometries can be considered since, whenever $h'(x)$ is a polynomial of degree m, the right-hand side of (3.6) can be repeatedly integrated by parts to arrive at a differential equation of degree $2[(m-1)/2]+2$ for the function whose $2[(m-1)/2]$th derivative is f. The highest-order derivatives come from the terms on the left-hand side of (3.6) and we therefore still expect that shocks will be possible when $|\lambda|$ and $|\kappa|$ are sufficiently small.

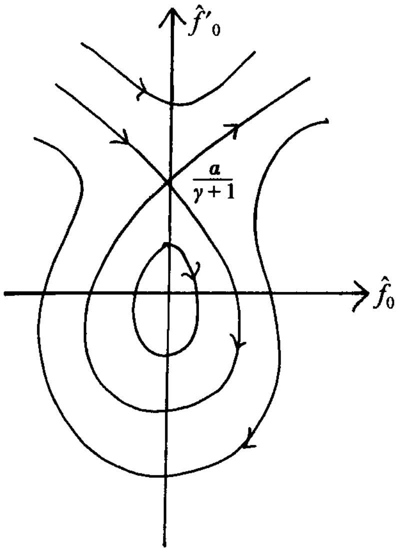

FIGURE 5. Phase plane for $\hat{f}_0, \hat{f}'_0$.

Keller (1977) used a Fourier series expansion of $h'(x)$ and $f'(x)$ to study (3.6). In particular for $h'(x) = \sin 2x$, he showed that the solution was very similar to the solution of (3.7) and in this case solutions contain shocks for λ between the values

$$\frac{16\kappa}{3\pi^2} \pm \frac{4}{\pi}\left[\frac{(\gamma+1)}{\pi}\left(1+\frac{16\kappa^2}{9\pi(\gamma+1)}\right)\right]^{\frac{1}{2}}.$$

More recently Chester (1993) has extended this work by considering

$$h(x) = A_1 \cos 2x + A_2 \cos 4x$$

and has obtained results very similar to those shown in figures 3 and 4.

The key to understanding the response of (2.12) when $\sigma = O(\epsilon^{\frac{1}{2}})$ lay in the derivation of equations such as (3.6) and (3.7) but this method is not available when σ is $O(1)$. We therefore now reconsider the case $\sigma = O(\epsilon^{\frac{1}{2}})$ under the assumptions that we could not solve the equation for ϕ_0 explicitly. We note that an alternative representation of the solution of (3.2) is as a linear combination of the eigenfunctions

$$\{\cos nx\,(a_n \cos nt + b_n \sin nt)\}_{n \in Z},$$

which are complete in a suitable function space, and then treat (3.3 *a*) by the Fredholm Alternative. This procedure has general applicability to the case when the operator in (3.2) does not have constant coefficients. We now fix ideas by considering the problem of solving

$$\phi_{1xx} - \phi_{1tt} = R(x, t) \tag{3.9a}$$

subject to the boundary conditions

$$\phi_{1x}(0, t) = 0, \quad \phi_{1x}(\pi, t) = \sin t, \quad \phi_1(x, t) = \phi_1(x, t+2\pi). \tag{3.9b}$$

Using the Fredholm Alternative on (3.9) leads to the unconventional infinite set of orthogonality conditions

$$\int_0^{2\pi}\int_0^{\pi} \cos nx(a_n \cos nt + b_n \sin nt)\, R(x, t)\,\mathrm{d}x\,\mathrm{d}t = -\pi b_1\, \delta_{1n} \tag{3.10}$$

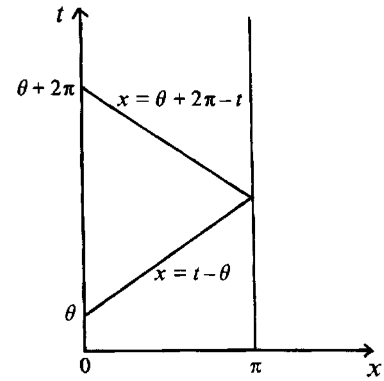

FIGURE 6. Contour for the integral in (3.13).

for any constants a_n, b_n and for $n = 0, 1, 2, \ldots$. In order to retrieve (3.6) from these conditions we note first that (3.10) implies that

$$\int_{\theta}^{2\pi+\theta}\int_{0}^{\pi} R(x,t)\cos nx \cos n(t-\theta)\,\mathrm{d}x\,\mathrm{d}t = -\pi \sin\theta\,\delta_{1n} \tag{3.11}$$

for any θ. Also, from the completeness of the relevant Sturm–Liouville problem,

$$\frac{1}{2}+\sum_{n=1}^{\infty}\cos nx\cos nT = \tfrac{1}{2}\pi\,[\delta(x-T)+\delta(x+T-2\pi)] \tag{3.12}$$

for $0 < x < \pi$, $0 < T < 2\pi$. Now summing (3.11) over n and using (3.12) yields

$$\int_{\Gamma} R(x,t)\,\mathrm{d}t = \int_{\theta}^{\theta+\pi} R(t-\theta,t)\,\mathrm{d}t + \int_{\theta+\pi}^{\theta+2\pi} R(\theta+2\pi-t,t)\,\mathrm{d}t = -2\sin\theta, \tag{3.13}$$

where Γ is the curve consisting of the characteristics $x = t-\theta$ and $x = \theta+2\pi-t$ in $0 < x < \pi$ as shown in figure 6. The result (3.13) is identical to (3.5) which was obtained by direct integration along the characteristics. Thus replacing R by the right-hand side of (3.3a) immediately leads to (3.6) with t replaced by θ. Conversely, it can be shown using Fourier series that (3.13) is equivalent to satisfying the conditions (3.10).

With this background we can now consider the more difficult parameter range where $\sigma \gg \epsilon^{\frac{1}{2}}$.

4. Strong geometric effects; the analysis of (2.12) when $\epsilon^{\frac{1}{2}} \ll \sigma$

4.1. $\epsilon^{\frac{1}{2}} \ll \sigma \ll 1$

In this region the geometric effects dominate the nonlinearity and the forcing which necessitates a more complicated expansion procedure than was used in §3. The situation that can arise now is illustrated by considering the case $\sigma = \epsilon^{\frac{1}{4}}\beta$ where $\beta = O(1)$, which means that we are assuming that $h_0^2/L^2 \ll O(\epsilon^{\frac{1}{4}})$. We proceed as before, writing $\phi = \phi_0+\epsilon^{\frac{1}{4}}\phi_1+\epsilon^{\frac{1}{2}}\phi_2+\ldots$ in (2.12) and deriving a sequence of problems for $\phi_0, \phi_1, \phi_2, \ldots$. Equation (3.2) still holds for ϕ_0 but now ϕ_1 and ϕ_2 satisfy

$$\phi_{1xx}-\phi_{1tt} = -\beta h'(x)\,\phi_{0x} \tag{4.1}$$

with $\phi_{1x} = 0$ at $x = 0, \pi$ and

$$\phi_{2xx}-\phi_{2tt} = \lambda\phi_{0tt}+(\gamma-1)\,\phi_{0t}\,\phi_{0tt}+2\phi_{0x}\,\phi_{0xt}-\beta h'(x)\,\phi_{1x}+\beta^2 h(x)\,h'(x)\,\phi_{0x} \tag{4.2}$$

with $\phi_{2x}(0,t) = 0$, $\phi_{2x}(\pi,t) = \sin t$.

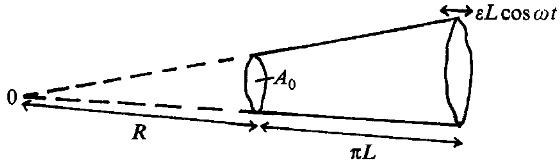

FIGURE 7. Frustrum of spherical annulus.

If we again write $\phi_0 = f(t-x)+f(t+x)$, we see immediately from (3.6) that (4.1) is only integrable if

$$\int_0^{\pi} ((h'(\pi-\tau)-h'(\tau))f'(t+2\tau)\,d\tau = 0, \tag{4.3a}$$

or, equivalently

$$\int_0^{\pi} h'(\tau)\,(f'(t-2\tau)-f'(t+2\tau))\,d\tau = 0. \tag{4.3b}$$

This leads to two possibilities; either $h'(x)$ is even about $x = \frac{1}{2}\pi$ and (4.3a) holds identically or else f has to satisfy certain constraints which we will discuss shortly. In either case, we can proceed by finding

$$\phi_1 = \beta \int_0^x f'(t-x+2\eta)\,(h(x-\eta)-h(\eta))\,d\eta + \sum_{n=1}^{\infty} (P_n \cos nt + Q_n \sin nt)\cos nx, \tag{4.4}$$

where P_n and Q_n are arbitrary and then the integrability condition for (4.2) implies that $f(t)$ must satisfy

$$\lambda f''+(\gamma+1)f'f''+\frac{1}{\pi}\sin t = \frac{\beta^2}{4\pi}\int_0^{\pi} f'(t+2\eta)\,(H(\eta)-H(\pi-\eta))\,d\eta$$
$$-\frac{\beta}{\pi}\sum_{n=1}^{\infty} n\gamma_n(P_n\cos nt + Q_n \sin nt), \tag{4.5}$$

where

$$H(\eta) = \int_0^{\pi-\eta} h'(\eta+\tau)\,h'(\tau)\,d\tau - 2h'(\eta)\,h(\eta)$$

and

$$\gamma_n = \int_0^{\pi} h'(x)\cos nx \sin nx\,dx.$$

When $h'(x)$ is even about $x = \frac{1}{2}\pi$, all the γ_n are zero and so (4.5) is of exactly the same form as (3.6) which was discussed in the previous section. This case is exemplified by a resonator consisting of a frustum of a slender cone cut off by two spheres of radius R and $R+L\pi$ oscillated near a resonant frequency with amplitude ϵL on one sphere (figure 7). If the distance from the centre of the spheres is $R(1+Lx/R)$, the cross-sectional area of the tube will be $A_0(1+Lx/R)^2$, and if $L/R \ll 1$ we can therefore identify L/R with σ and $2x+\sigma x^2$ with $h(x)$. Since $h'(x)$ is an even function in $(0,\pi)$ to order σ, the response when $\sigma = O(\epsilon^{\frac{1}{2}})$ is unaffected by the varying cross-section. However, when $\sigma = \epsilon^{\frac{1}{4}}\beta$ we need to write $h = h_0+\epsilon^{\frac{1}{4}}h_1$ with $h_0 = 2x$ and $h_1 = \beta x^2$, and then using the above analysis we are led to equation (4.5) with $H(\eta)$ replaced by $H_0(\eta)+(2/\beta)\,h_1'(\eta)$. Then the right-hand side reduces to $-2\beta^2 f(t)$ and so the equation for f is identical to (3.8) with κ replaced by $-2\beta^2$. Thus we see that for this problem the change from shock waves to a single-mode response occurs when $L/R = O(\epsilon^{\frac{1}{4}})$.

For more general h we can use the Fredholm Alternative approach, noting that

$$\phi_0 = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \cos nx$$

and that the integrability condition for (4.1) implies that

$$a_n \gamma_n = b_n \gamma_n = 0, \quad n = 1, 2, \ldots, \tag{4.6}$$

and these conditions (4.6) are equivalent to condition (4.3). Consideration of the left-hand side of (2.12) now shows that the eigenfrequencies are

$$\omega_n = n\left(1 + \frac{\beta\epsilon^{\frac{1}{4}}\gamma_n}{\pi n} + O(\epsilon^{\frac{1}{2}})\right) \tag{4.7}$$

so that γ_n is a measure of the non-commensurability of the nth natural frequency of the linear system with the forcing frequency. We now consider the situation when some or all of the Fourier coefficients γ_n are non-zero. For some of our discussion it will be helpful to refer back to the response in figure 4 for large κ; there the area variation, although of a smaller order of magnitude than that considered here, is such that the γ_n are all non-zero.

The sensitivity of the dependence of a response to γ_n can be seen by considering cases where γ_n are non-zero for small values of n.

(*a*) $\gamma_1 \neq 0$

In this case the area variation has shifted ω_1 by $O(\epsilon^{\frac{1}{4}})$ and so, within the detuning region we have selected, there is no resonant amplification. We find that $\phi_0 = 0$ is a solution regardless of the values of γ_i for $i > 1$ and there is a single-mode response,

$$\bar{\phi} = -\frac{\epsilon^{\frac{1}{4}}}{\beta\gamma_1} \cos x \sin t,$$

which matches with the solution of (3.8) as κ increases with $\lambda = O(1)$. To obtain a resonant response with $\bar{\phi} = O(1)$ we would need to replace λ by $2\beta\epsilon^{-\frac{1}{4}}\gamma_1/\pi + \lambda$; this would be equivalent to the case $\gamma_1 = 0$ and we shall assume $\gamma_1 = 0$ henceforth.

(*b*) $\gamma_1 = 0$, $\gamma_2 \neq 0$

In this case $\bar{\phi}$ is $O(1)$ but conditions (4.6) imply that $a_2 = b_2 = 0$. It is still possible to write $\bar{\phi} = f(t-x) + f(t+x)$ as long as f satisfies the constraints

$$\int_0^{2\pi} f(x) \cos 2x \, dx = \int_0^{2\pi} f(x) \sin 2x \, dx = 0$$

and we can again proceed to the integrability condition for ϕ_2 and obtain (4.5). Now we can see that there is a solution of the form

$$f(t) = A \sin t,$$

with
$$P_2 = 0, \quad Q_2 = \pi(\gamma + 1) A^2 / 4\beta\gamma_2$$
and
$$\gamma_i P_i = \gamma_i Q_i = 0 \quad \text{if} \quad i > 2,$$
where
$$A = \left[\lambda\pi - \frac{\beta^2}{2} \int_0^{\pi} H(\eta) \sin 2\eta \, d\eta\right]^{-1},$$

and this solution remains valid regardless of the values of γ_i for $i > 2$.

Thus the non-vanishing of γ_2 has the dramatic effect of eliciting a single-mode response as long as λ is away from a narrow detuning band near

$$\lambda = (\beta^2/2\pi)\int_0^\pi H(\eta)\sin 2\eta\,d\eta.$$

Chester (1993) analysed $h(x) = k_2\cos 4x$ in (3.6) and found a finite shock regime as $\kappa \to \infty$, indicating that there will be an $O(1)$ range of λ near $\lambda = 0$ for which shocks exist when $\sigma = O(\epsilon^{\frac{1}{2}})$, and we hypothesize that in general the range of κ in which there are shocks decreases as σ increases. As we shall see later we expect the shocks to have vanished completely by the time $\sigma = O(1)$.

In addition, figure 3 indicates that there will be other 'resonances' of a Duffing type when $\beta = O(1)$. These are, however, not described by equation (4.5) since they do not lie within the detuning band where $\lambda = O(1)$.

(*c*) $\gamma_1 = \gamma_2 = \ldots = 0, \gamma_N = 0, \gamma_{N+1} \neq 0,\ \gamma_{N+2} \neq 0 \ldots \gamma_{2N} \neq 0$

The above argument can be extended to cover this case but we will not present the details here; we merely note that f must satisfy the constraints

$$\int_0^{2\pi} f(x)\cos nx\,dx = \int_0^{2\pi} f(x)\sin nx\,dx = 0$$

for all n for which $\gamma_n \neq 0$ as well as satisfying (4.5). Then the fact that P_n and Q_n are unknown gives us enough flexibility to find both $f(t)$ and P_n, Q_n for values of n for which $\gamma_n \neq 0$. We are able to find a response in the form of a linear combination of discrete modes with frequencies $(1, 2, \ldots, N)$ except for narrow ranges of values of λ where shock solutions are still possible.

The picture that emerges is that in this parameter regime the response resembles that described for $\sigma = O(\epsilon^{\frac{1}{2}})$ when $h'(x)$ is even about $\frac{1}{2}\pi$ (i.e. when $\gamma_n = 0$ for all n). As soon as this condition is not satisfied and γ_n is non-zero for $n > N$, we expect the solution to consist of a discrete mode response containing the first N frequencies except for small 'detuning bands' in the neighbourhood of which shocks may occur. These detuning bands occur along the extensions of the 'fingers' in the κ–λ diagram (figure 3) and are associated with Duffing-like responses for each non-zero γ_n. The amplitude of the resonance grows and the shock bands appear to decrease as σ increases in size.

We note that for $1 \gg \sigma \gg O(\epsilon^{\frac{1}{4}})$, complicated responses similar to these described above are likely to develop unless $h(x)$ has a very special form. Indeed, $h'(x)$ not only needs to be even itself but also needs to be such that $H(x)$ is even in $(0, \pi)$ if we are to retain the Chester response when $\sigma = O(\epsilon^{\frac{1}{4}})$. Then this process is exacerbated as σ increases through $\sigma = O(\epsilon^{\frac{1}{6}})$, $O(\epsilon^{\frac{1}{8}})$ etc, leading to progressively more and more restrictions on $h(x)$ if we are to retain the Chester response to lowest order.

4.2. $\sigma = O(1)$ *and* $h_0^2/L^2 \ll \epsilon^{\frac{1}{2}}$

This is the regime where area variations are large enough to affect the linearized spectrum. When we write $b(x) = \sigma h'(x)/(1+\sigma h(x))$, $\bar{\phi} = \phi_0 + \epsilon^{\frac{1}{2}}\phi_1 + \ldots$, we then need to solve the first-order problem

$$\phi_{0xx} + b(x)\,\phi_{0x} - \phi_{0tt} = 0, \tag{4.8a}$$

with $\phi_{0x} = 0$ at $x = 0, \pi$. The solutions of this problem take the form

$$g(x)\,(A\cos\omega t + B\sin\omega t)$$

where

$$g''+bg'+\omega^2 g = 0, \tag{4.8b}$$

and our earlier work leads us to expect that only normal modes for which ω is an integer will be excited by the forcing. Thus the response now depends on the number of normal frequencies commensurate with the forcing frequency. Guided by our earlier work we anticipate that the response will depend sensitively on the commensurability of the eigenfrequencies ω_n with the forcing frequency and we sketch out the possible scenarios.

(*a*) $\omega_1 \neq 1$

Here we expect that $\bar{\phi}$ will have amplitude of $O(\epsilon^{\frac{1}{2}})$ and there will be no resonant amplification in this detuning band.

(*b*) $\omega_1 = 1,\ \omega_n \neq n,\ n > 1$

This case has been considered by Can & Askar (1990) and Ellermeier (1993). There is a single-mode response whose amplitude depends on λ as in a Duffing equation. The possibility of shock solutions has now completely disappeared unlike (*b*) in §4.1 above.

(*c*) $\omega_n = n$ for a finite number N of values of n

Generalization of the work in (*b*) has been carried out by Peake (1993) for $N = 2$, and 3; he shows that the response consists of a finite system of N coupled Duffing-like algebraic equations for the amplitudes of the resonating modes, again with no possibility of shocks occurring.

(*d*) $\omega_n = n$ for all n

There are well-established procedures for retrieving $b(x)$ knowing the spectrum ω_n (see for example Barcilon 1983) and it is easy to show that one form of $b(x)$ which gives $\omega_n = n$ is $b = -2\alpha/(\alpha x+1)$ where α is constant. For this case, which is equivalent to $1+\sigma h = (1+\alpha x)^{-2}$, Keller (1977) found that (4.8) has the general solution

$$\phi_0 = (1+\alpha x)(G'(t+x)+G'(t-x))+\alpha(G(t-x)-G(t+x)),$$

where G is any 2π-periodic function which can be shown, on going to the second term in the expansion, to satisfy Chester's equation. Thus shock solutions are still possible for $\sigma = O(1)$ when all the eigenfrequencies are integer multiples of the forcing frequency. It has not yet been established that this particular form of $b(x)$ emerges as the limit of the sequence of constraints on $h(x)$ that we discussed at the end of §4.1.

Our conclusions from the above evidence are that if $\omega_n - n \leqslant O(\epsilon^{\frac{1}{2}})$ for all n, there will be a shock regime whereas if $\omega_n - n \gg O(\epsilon^{\frac{1}{2}})$ for any n, there will be no shocks to first order. Geometrically this is equivalent to saying that for $\sigma \leqslant O(\epsilon^{\frac{1}{2}})$ shocks are always possible but for $\sigma \gg O(\epsilon^{\frac{1}{2}})$ shocks will only occur if the geometric variations satisfy restrictions that become increasingly severe as σ increases to $O(1)$.

5. Conclusion

We have attempted to present a theoretical framework to describe the general way in which the nonlinear response of nearly one-dimensional acoustic resonators changes as their area imperfections increase. Dissipation has been neglected throughout and its inclusion is clearly a topic for further research.

The general picture is that while perfectly one-dimensional resonators always have a detuning band within which shock waves occur, increasing the imperfection beyond a certain threshold level suppresses the shocks for all but a few special 'tuned' geometries. For $\sigma \ll O(\epsilon^{\frac{1}{2}})$, the response is the classical one of Chester (1964). For $\sigma \sim \epsilon^{\frac{1}{2}}$, there is a wide variety of behaviour depending on the resonator shape. Most geometries yield threshold values of $\sigma\epsilon^{-\frac{1}{2}}$ beyond which shocks are not possible, but there are rare geometries where shocks can still occur for quite large values of $\sigma\epsilon^{-\frac{1}{2}}$. For $\sigma \gg \epsilon^{\frac{1}{2}}$, the situation becomes more and more intricate because, depending on the behaviour of certain Fourier coefficients, we may be confronted with a very difficult problem involving an infinite number of Fredholm alternatives.

We can comment further that if a resonator has symmetries that enable the general linear symmetric response to be written down explicitly, we can, in principle, always make progress towards reducing the problem to that of an ordinary differential equation. In the previous Section, we have noted the problem posed by Keller, where area variation proportional to $1/(1+\alpha x)^2$ permitted a response with shocks even though $\sigma = O(1)$. This happened because the (-1)th dimensional symmetric wave equation is explicitly soluble in a form which is convenient for further manipulations and because all the natural frequencies are commensurate. Although other area imperfections leading to a $(2n+1)$-dimensional wave equation can be solved explicitly, in particular the spherical annular geometry considered by Peake (1993), the natural frequencies are only commensurate with the forcing when $n = 1$ or -1, so this remains a very special case. We also note that, for an organ pipe that is open to the atmosphere at one end, so that the pressure vanishes there, the spectrum consists of the odd integers and an analysis of (3.1) leads to a real functional differential equation for f (Chester 1981).

We remark that all our work has been based on the assumption that any shocks which occur must be weak enough not to affect the homentropy of the flow to the order we have been considering. Since our dimensionless velocities relative to the sound speed have been at most of $O(\epsilon^{\frac{1}{2}})$, any expansions we may use involving shock wave responses may not be valid to $O(\epsilon^{\frac{3}{2}})$. None of the expansions in this paper have needed terms of this order, but entropy changes have been considered in more detail by Keller (1976*b*).

Finally we note one intriguing open question concerning the unforced nonlinear oscillations of these resonators. For a finite-dimensional system such as Duffing's equation, the free response can usually be deduced from the forced response by letting the forcing amplitude tend to zero; in Duffing's equation this yields the result that free oscillations with detuning of $O(\lambda)$ have amplitude of $O(\lambda^{\frac{1}{2}})$. However, to the order to which we have been working, the perfect organ pipe has no free modes in the presence of nonlinearity because the response to a forcing amplitude ϵ is proportional to $\epsilon^{-\frac{1}{2}}$ no matter what the detuning λ happens to be. Hence the presence of increasing geometrical imperfections, which tend to make the resonator behave more and more like a finite degree of freedom oscillator, may be crucial for a free response to be possible, but the few shapes for which shocks occur in the forced response may not be capable of sustaining such a free response even if $\sigma = O(1)$.

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